4.3 Projectile Motion

Projectile Motion

Key Ideas

- Projectile motion is the motion of an object subject only to the acceleration due to gravity, which is constant near the surface of Earth.
- To solve projectile motion problems, we analyze the motion of the projectile in the horizontal and vertical directions using the kinematic equations for x and y.

Learning Objectives

After completing this section, you should be able to...

- use one-dimensional motion in perpendicular directions to analyze projectile motion,
- calculate the range, time of flight, and maximum height of a projectile that is launched and impacts a flat, horizontal surface,
- find the time of flight and impact velocity of a projectile that lands at a different height from that of launch, and
- calculate the trajectory of a projectile.

Projectile motion is the motion of an object thrown or projected into the air, subject only to acceleration as a result of gravity and ignoring the frictional effects of air drag. The applications of projectile motion in physics and engineering are numerous. Some examples include meteors as they enter Earth's atmosphere, fireworks, and the motion of any ball in sports. Such objects are called **projectiles** and their path is called a **trajectory**. The motion of falling objects was discussed in the previous chapter (Motion Along a Straight Line). Such motion, where there is no horizontal movement, can be thought of as a one-dimensional type of projectile motion. In this section, we consider two-dimensional projectile motion, ignoring the effects of air resistance. As such, it should be remembered that projectile motion is a **model**, and *it only effectively approximates motion when air resistance is relatively small.* It would be a poor model for describing the motion of a thrown paper napkin, for example.

The most important fact to remember here is that motions along perpendicular axes are independent and thus can be analyzed separately. We discussed this fact in Displacement and Velocity, where we saw that vertical and horizontal motions are independent. The key to analyzing two-dimensional projectile motion is to break it into two motions: one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible because acceleration resulting from gravity is vertical; thus, there is no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the x-axis and the vertical axis the y-axis. It is not required that we use this choice of axes; it is simply convenient in the case of gravitational acceleration. In other cases we may choose a different set of axes. The following figure illustrates the notation for displacement, where we define $\vec{d}$ to be the total displacement, and $\vec{x}$ and $\vec{y}$ are its component vectors along the horizontal and vertical axes, respectively. When working with the equations of kinematics, the variables are usually written without the vector symbols, so we must be aware that we are dealing with vectors that have both magnitude and direction.

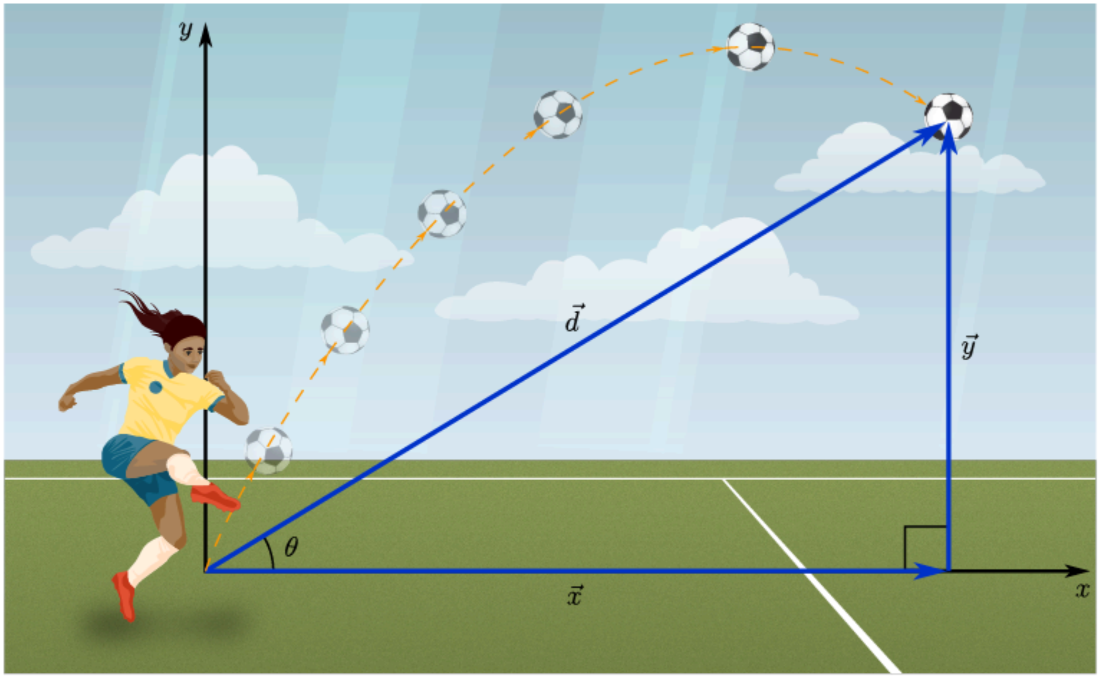

Figure 4.13 The position of a kicked soccer ball is shown at various points along its trajectory. At the rightmost position, we show the displacement vector and its x- and y-components. The angle θ of the displacement vector with respect to the $+x$-direction can be found using trigonometry.

To describe projectile motion completely, we must include velocity and acceleration, as well as displacement. Defining the positive direction to be upward, the components of acceleration are then very simple:

$$a_x = 0$$
$$a_y = -g = -9.8 m/s^2$$

This means the initial velocity in the x-direction is equal to the final velocity in that direction, or $v_{x,f} = v_{x,i}$. With these conditions on acceleration and velocity, the kinematic equations for a constant acceleration from Motion with Constant Acceleration can be used and are somewhat simplified. Using this set of equations, we can analyze projectile motion, keeping in mind some important points.

Problem-Solving Strategy

Projectile Motion

1. Resolve the motion into horizontal and vertical components along the x- and y-axes. The vector components of displacement along these axes are written as x and y without the vector symbol. Likewise, the components of the velocity v are $v_x = v\cos\theta$ and $v_y = v\sin\theta$.
2. Treat the motion as two independent one-dimensional motions: one horizontal and the other vertical. Use the kinematic equations for horizontal and vertical motion presented earlier.
3. Solve for the unknowns in the two separate motions: one horizontal and one vertical. Note that the only common variable between the motions is the elapsed time t. The problem-solving procedures here are the same as those for one-dimensional kinematics and are illustrated in the following solved examples.
4. Recombine quantities in the horizontal and vertical directions to find the total displacement $\vec{d}$ and velocity $\vec{v}$. Solve for the magnitude and direction of the displacement and velocity using $d = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, and $v = \sqrt{v_x^2 + v_y^2}$, where θ is the direction of the displacement.

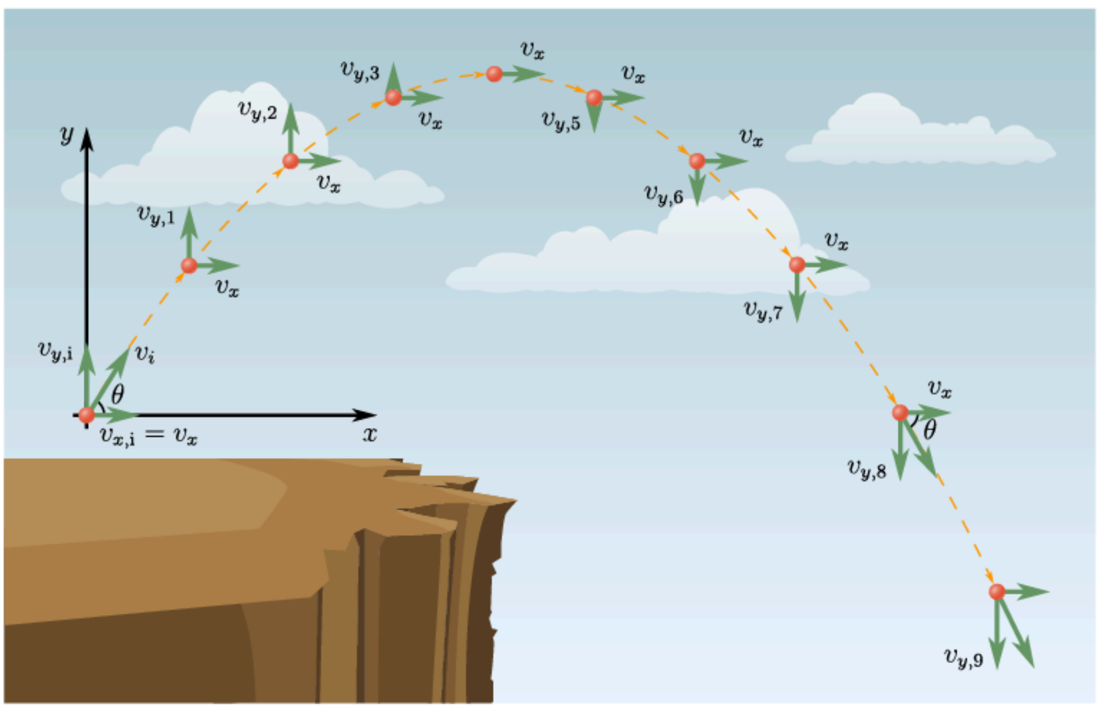

Figure 4.14 We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. The horizontal motion is simple, because the x-component of the velocity is constant since there is no acceleration in that direction. The velocity in the vertical direction begins to decrease as the object rises. At its highest point, the vertical component of the velocity is zero. As the object falls toward Earth again, the vertical component of the velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity.

Example 4.7

Fireworks

During a fireworks display, a shell is shot into the air with an initial speed $v_1 = 70.0\ m/s$ at an angle $\theta = 75^\circ$ above the horizontal, as illustrated in Figure 4.13. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passes between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes? (d) What is the total displacement from the point of launch to the highest point?

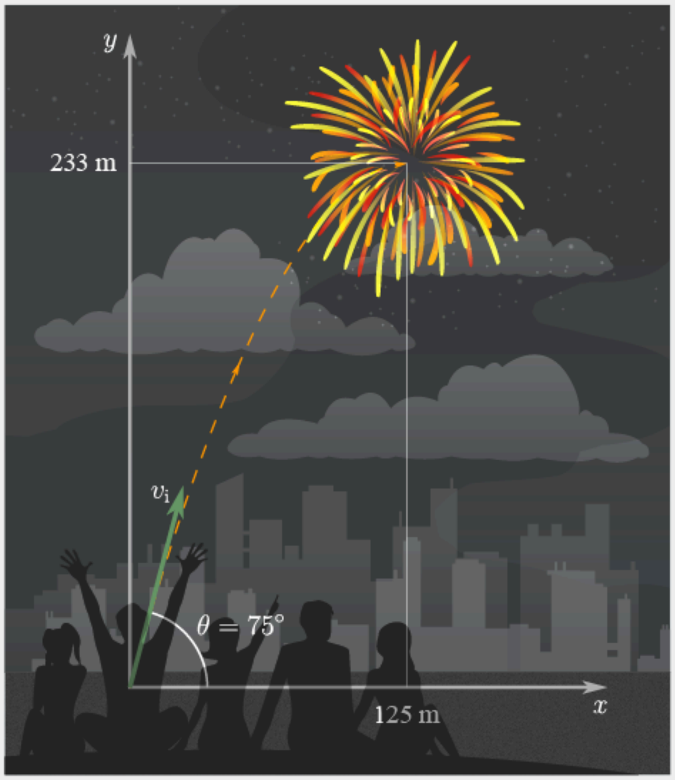

Figure 4.15 The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, found at a height of 233 m and 125 m away horizontally.

Strategize

The motion can be separated into horizontal and vertical motions in which $a_x = 0 m/s^2$ and $a_y = -g$. We can then define both initial position coordinates of the shell, x_i and y_i, to be zero. At the highest point, or apex, of the trajectory, $v_y = 0 m/s$. Given this information, we'll select an appropriate kinematic equation to complete part (a). For part (b), we can find the elapsed time at which the final vertical speed is zero m/s. The horizontal motion is happening at the same time as for the vertical motion, so we can find the horizontal displacement using the time found in part (c), since the horizontal component of the velocity is known. Having found the vertical and horizontal components of the displacement, we can then find the magnitude and direction of the displacement of the shell when it explodes.

Develop and Solve

(a) Using our strategy for part (a), the appropriate kinematic equation is $v_{y,f}^2 = v_{y,i}^2 + 2a_y(y_f - y_i)$. With the initial height and final vertical velocity component equal to zero, we have

$$0 = v_{y,i}^2 - 2gy_f$$

Solving for y and inserting the appropriate expression for the initial vertical component of the velocity allows us to find the final expressions for the final height and substitute the given values.

$$\begin{aligned} y_f &= \frac{v_{y,i}^2}{2g} \\ &= \frac{(v_i \sin(\theta))^2}{2g} \\ &= \frac{(70.0 \text{ m/s } (\sin 75^\circ))^2}{2(9.80 \text{ m/s}^2)} \\ &= 233 \text{ m} \end{aligned}$$

(b) As in many physics problems, there is more than one way to solve this. To find the time the projectile reaches its highest point in this case, the easiest method is to use $v_{y,f} = v_{y,i} + a_y t$. Because $v_{y,f} = 0 \text{m/s}$ at the apex, this equation is simplified. We can now solve for the elapsed time.

$$\begin{aligned} 0 &= v_{y,i} - gt \\ t &= \frac{v_{y,i}}{g} \\ &= \frac{v_i \sin(\theta)}{g} \\ &= \frac{(70.0\text{m/s}) \sin(75^\circ)}{9.80\text{m/s}^2} \\ &= 6.90\text{s} \end{aligned}$$

(c) Because we are told air resistance is negligible, the horizontal component of the shell's acceleration $a_x = 0\text{m/s}^2$, and the horizontal velocity is constant. The horizontal displacement is then the horizontal velocity multiplied by the elapsed time, $x_f = x_i + v_x t$, where is the x-component of the initial velocity of the shell.

$$\begin{aligned} x_f &= v_i \cos(\theta) t \\ &= (70.0\text{m/s})(\cos 75^\circ)(6.90\text{s}) \\ &= 125\text{m} \end{aligned}$$

(d) The horizontal and vertical components of the displacement have been determined, so all that is needed here is to find the magnitude and direction of the displacement at the highest point: